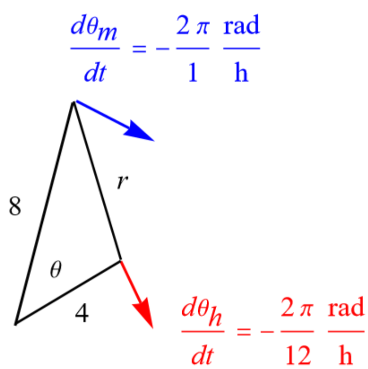
Exercise 50

The minute hand on a watch is 8 mm long and the hour hand is 4 mm long. How fast is the distance between the tips of the hands changing at one o'clock?

Solution

Draw a schematic of the watch at a certain time.



The aim is to find dr/dt when $\theta = \pi/6$; this is from the fact that there are 12 hours every 2π radians, so after the first hour has passed the angle is $2\pi/12 = \pi/6$. Start with the formula relating the sides of this triangle, the law of cosines.

$$r^2 = 4^2 + 8^2 - 2(4)(8)\cos\theta$$

$$= 80 - 64\cos\theta$$

$$r = \sqrt{80 - 64\cos\theta}$$

Take the derivative of both sides with respect to time by using the chain rule.

$$\frac{d}{dt}(r) = \frac{d}{dt}\left(\sqrt{80 - 64\cos\theta}\right)$$

$$\frac{dr}{dt} = \frac{1}{2}(80 - 64\cos\theta)^{-1/2} \cdot \frac{d}{dt}(80 - 64\cos\theta)$$

$$= \frac{1}{2}(80 - 64\cos\theta)^{-1/2} \cdot \left[-64(-\sin\theta) \cdot \frac{d\theta}{dt}\right]$$

$$= \frac{32\sin\theta}{\sqrt{80 - 64\cos\theta}}\frac{d\theta}{dt}$$

$$= \frac{32\sin\theta}{\sqrt{80 - 64\cos\theta}}\frac{d}{dt}(\theta_m - \theta_h)$$

$$= \frac{32\sin\theta}{\sqrt{80 - 64\cos\theta}}\left(\frac{d\theta_m}{dt} - \frac{d\theta_h}{dt}\right)$$

Plug in the values for the angular velocities of the minute and hour hands.

$$\frac{dr}{dt} = \frac{32\sin\theta}{\sqrt{80 - 64\cos\theta}}\left[\left(-\frac{2\pi}{1}\right) - \left(-\frac{2\pi}{12}\right)\right]$$

$$= \frac{32\sin\theta}{\sqrt{80 - 64\cos\theta}}\left(-\frac{11\pi}{6}\right)$$

Therefore, when it's one o'clock, the rate of change of the distance between the minute- and hour-hand tips with respect to time is

$$\left.\frac{dr}{dt}\right|_{\theta=\pi/6} = \frac{32\sin\left(\frac{\pi}{6}\right)}{\sqrt{80 - 64\cos\left(\frac{\pi}{6}\right)}}\left(-\frac{11\pi}{6}\right) = -\frac{22\pi}{3\sqrt{5 - 2\sqrt{3}}}\ \frac{\text{mm}}{\text{h}} \approx -18.5896\ \frac{\text{mm}}{\text{h}}.$$